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A SENSE OF SPACE

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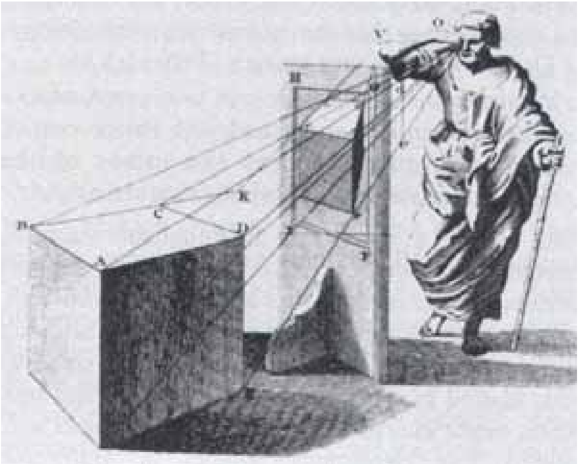
A recent exhibition at the Wilhelm-Hack-Museum in Ludwigshafen am Rhein, West Germany, was devoted to original works of recognized international artists working in the field of constructive and concrete art. The aim of the exhibition was to give a comprehensive survey of the relationships between mathematics and art in the last three decades and to trace possibilities of linking the two subjects in the classroom.

Constructive art makes use of mathematical principles. When composing a painting or a sculpture, the artist employs a System based on a geometrical or arithmetical concept and then creates either a single piece of work or a series which shows various aspects or a progression of that concept. Ordered systems of lines and shapes, the application of modules and grids and, of course, the rational use of colour, which takes into account the science of colour, are all mediums for expressing emotion and rhythm.

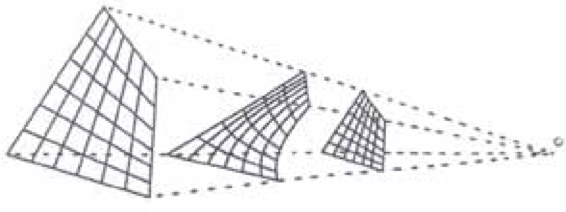
The topics for activities are endless: patterns, relations, matrices, groups, geometrical transformations, shapes, symmetry, angles, 2-D representations of 3-D space, measurement, the use of Computers, links to other subjects — in particular to music. For the teaching and learning of mathematics constructive art provides an exciting and innovating field for investigation ad infinitum.

In the following 1 shall give a brief outline of two dimensional representations of three-dimensional space paying particular attention to perspective. Historically, this aspect is the most important one as it links modern and traditional art and shows the role mathematics has played in art over the last centuries. Our knowledge of linear perspective goes back to a development which started in Florence in the early l5th century when new demands on the pictorial telling of stories were made. The audience of these times was no longer satisfied with the hieratic composition of symbols but welcomed the new form of representation which was painted as if the artist himself was an eye-witness of the happening.
still being debated. With the help of Euclidean geometry they certainly would have been able to do so.

The construction of spatial pictures by means of perspective leads to an ideal space which does not exist in reality. The principle of linear perspective is based on the physical law that light propagates rectilinearly in a homogenous medium, and that consequently our view is normally confined to straight lines. The mathematical construction is defined by the central projection of the objects to be represented upon the plane of the picture. [1]

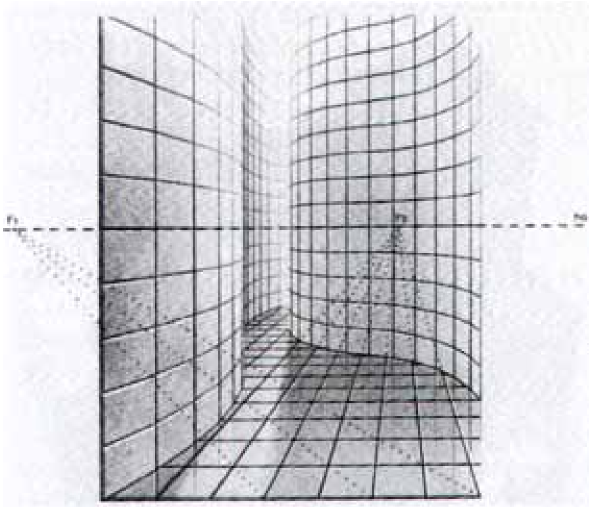


The centre of projection 0 is the point in the spectator‘s eye which is the apex of the visual angles. The two-dimensional representation of the object is the cross-section of the visual cone or pyramid of sight. it is important to note that although the central projection is a one-to-one transformation, linear perspective is not reversible: there are an infinite number of equivalent configurations which reproduce the same image. [2]

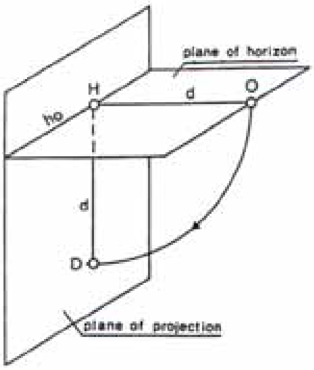


When drawing in linear perspective, the artist usually applies certain rules which are derived from
There is no doubt that the ancient Greeks already dealt with 2-D representations. Whether they were the original discoverers of the laws of perspective is the mathematical theory. Here are the most important ones (illustrated in pictures by Hans Peter Reuter shown above and on p3O):

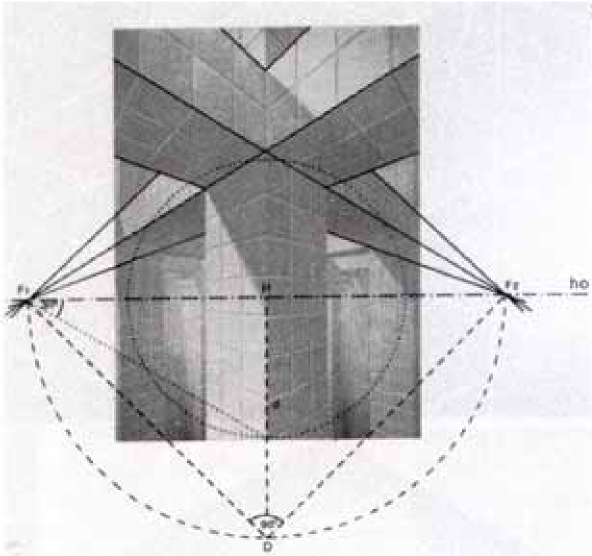
- Sets of parallel lines (like some horizontals, and all verticals) which are parallel to the plane of the picture remain parallel. Equal intervals on such lines remain equal.
- Sets of parallel lines which are not parallel to the picture-plane become lines converging on a ‚focal point‘. For horizontals, the focal points (F1 and F2) will be on the, horizon‘ h0. Equal intervals on such lines will not remain equal.



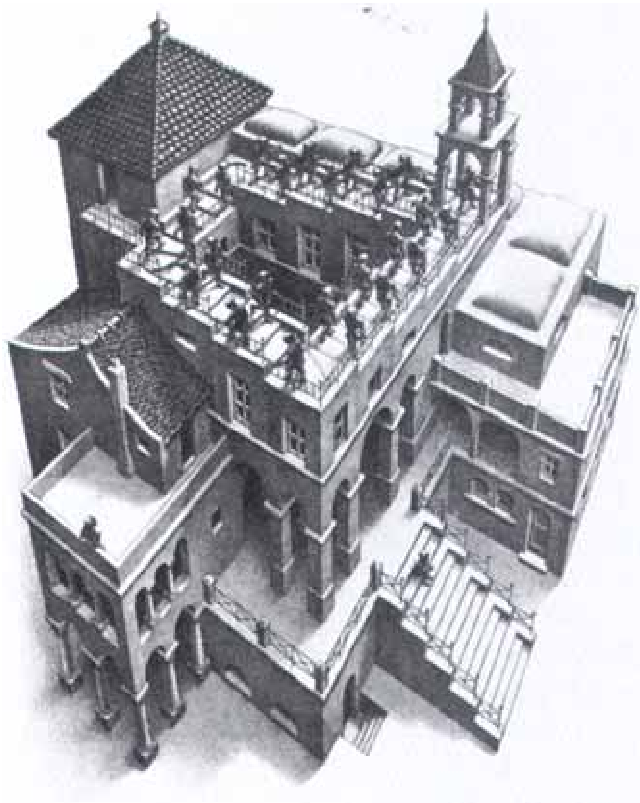
The measurements of an object drawn in perspective can be determined by a reconstruction. In order to do this an “inner orientation” of a picture is determined by the ‚horizon‘ h0, the ‚main point‘ H, the orthogonal distance d and the ‚distance point‘ D. Once these are set, the plan and elevation of an object can be constructed. If the length of the side of an object is known, the scale of the picture can be determined. This is called the, outer orientation‘.

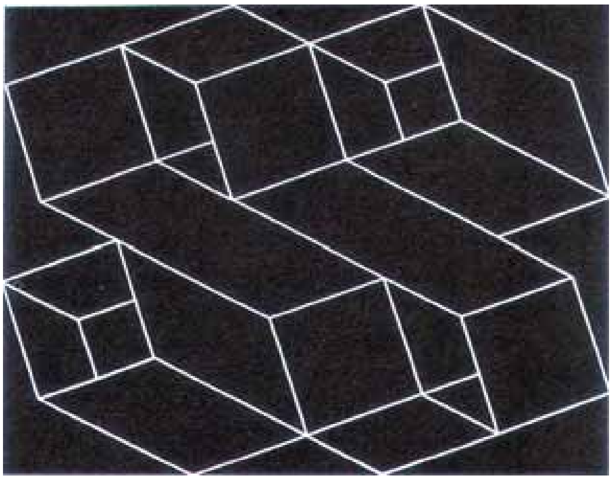
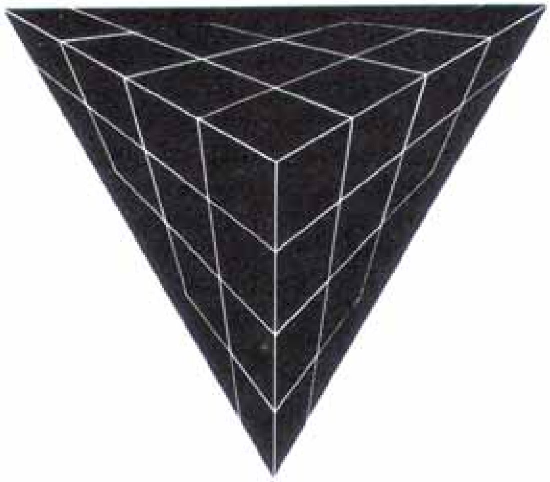
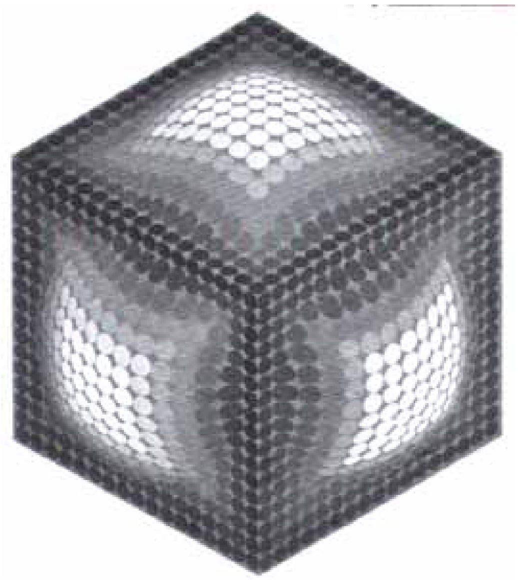
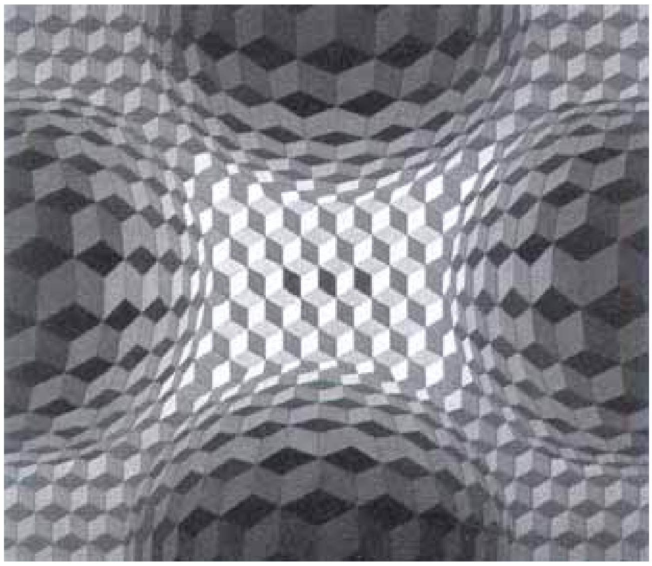


Modern artists often do not employ linear perspective in a mathematically correct manner in order to give the picture a certain tension, an impact or an instability. Well known examples of this are the works of the surrealists like Dali, Magritte and Chirico. Inner orientation is a valuable tool for analyzing paintings, drawings and photographs and provides the key 10 a deeper understanding of works of modern art as in the following case.
Hans Peter Reuter <born 1942 in Schwenningen, lives in Karlsruhe) is well known for his paintings of tiled rooms and public baths. His rooms are usually painted in shades of blue and are of plain functionality and strict order without a trace of persons or objects — for example, the picture shown on p3O. His most recent oil-painting light-room (399) (see below) shows the perspective view of a symmetrical interior or exterior, apparently drawn correctly, where the shadows of the asymmetrical incidence of light emphasize the spatial illusion. The inner orientation of this picture reveals, however, a distance d of only 120 cm. This value is not large enough to perceive the picture as a whole without seeing strong distortions and consequently the circle of sight covers only about half the area of the picture. Everybody who views this large painting (200 x 150 cm) will automatically step back. But then his eye is no longer in the position of the eye-point 0 determined by the central projection. The crossbeams appear heavy and threatening. Reuter deliberately employs a slightly incorrect use of perspective in order to dramatize the architecture of his picture and to induce a certain element of atmosphere. The spectator feels the heavy tension between the threatening cross-beam, the depth of the missing floor and the maze of the exits.



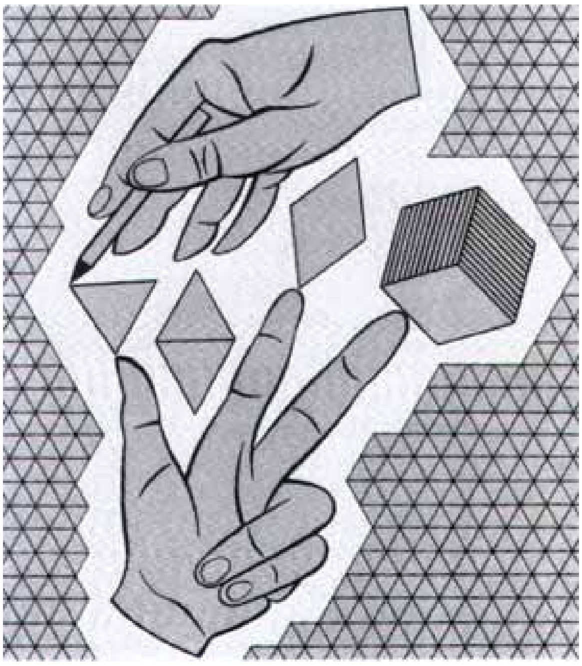
Spatial illusion has always played an important role in painting, ever since the principles of linear perspective had been discovered. For a comprehensive review of this matter the reader may consult the important books by Pirenne [3] and Gombrich [4]. Psychological experiments on perception have shown that our perceptions do not reveal the reality but pick out, ie ‚see‘, these things that appear to us most likely or most simple and that we already know. Renaissance artists achieved the spatial illusion of depth by painting tiled floors and using the expectations of the spectator who assumes that the floor is fiat and that all tiles have the same size and shape and who interprets their progressive reduction and foreshortening as spatial depth. Similarly, the painted ceilings of the Baroque constitute, with their visual ambiguity, the perfection of spatial illusion and of the masterly ‚trompe l‘oeil‘ where the spectator projects his own expectations onto the picture.



On the other hand, visual ambiguity has also been employed to demonstrate wrong perspective. Maurits Gomelis Escher (1898 — 1972, Holland), who is well known to mathematicians but difficult to place amongst his contemporaries, examined the principles of perspective in great detail. By applying them to unusual situations he found unexpected extensions. In a series of lithographs he produced various conflict situations and designed impossible worlds which cannot exist in 3-D space. One of his masterpieces is his work Ascending and descending (see p3O). Here the flight of stairs, which is drawn correctly in a horizontal plane, is overlapped by a spiral movement leading upwards. The optical illusion becomes apparent when we dissect the staircase horizontally into slices and find that individual stairs are not situated in horizontal planes.

An alternative way of transforming 3-D space into two dimensions are orthogonal projections where the projection lines meet the plane of projection in a right angle. The works of Joseph Albers (1888 Westphalia — 1976 New Haven, Connecticut) fall into this category. In his series ‚structural constellations‘ (an example is shown on p3O) he deals with hollow spaces and cavities and uses the straight line as a basis for a new visual system, the ‚skeleton space‘. A prominent feature of his constructions is the conflict between plane and spatial interpretation. By seeing the continuous alternating of the structure and flipping over of the shapes, the spectator becomes aware of his own perception.

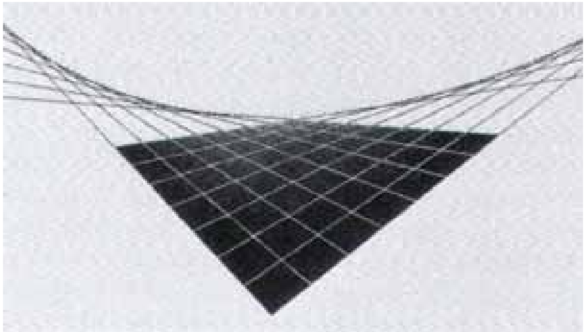


Victor Vasarely (born 1908 in Hungary, lives in Paris) is one of the few artists who systematically look for practical solutions in the visual shaping of our environment. His creations and decorations in the architectural field include geometrical structures which produce kinetic-optical effects and are based on standardisable modules. An important example is the ‘Kepler-cube‘ which consists of three congruent rhombi and is one of the fundamental structures of kineticism. The drawing below shows how the hands of the artist create this cube from the equilateral triangle [5]; two applications are shown on p3O. Vasarely uses his modules to create objective, geometrical structures with strong spatial illusions where, as in the work of Albers, the effect of flipping over of the diamonds stimulates alternative ways of viewing the picture.
2-D representations of space which employ neither central nor orthogonal projection are used by Hansjörg Glattfelder (born 1937 near Zurich, lives in Milano). His approach is highly intellectual and is based on the fact that, strictly speaking, the physical space in which we live does not have the properties of Euclidean geometry.

In the early l9th century, Gauss was one of the first mathematicians to consider the possibility of the existence of non-Euclidean geometries. His aim was to find out whether the sum of the three angles of a triangle is equal to two right-angles, in accordance with Euclidean geometry, or whether this sum differs from 180° as should be the case if physical space is curved and non-Euclidean. Gauss set out to measure the angles of a triangle defined by the tops of three mountains and having sides measuring 69, 85 and 107 km. The results of his measurements failed, however, to indicate any significant deviation from 180° outside the limits of experimental error [6].

In our century Albert Einstein could show in his general theory of relativity that in the presence of a gravitational field we cannot use Euclidean geometry because light does not travel in straight lines. In the vicinity of the earth, however, this effeet is so weak that the deviations of light rays from straight lines can be neglected for all practical purposes.

On the other hand, we arc usually unaware of the fact that we automatically interpet our perceptions in terms of Euclidean space. In our development of spatial concepts, the construction of Euclidean space presupposes a spatial system of coordinates, which implies that the orientation of the objects and their movements in space can be determined by relations like left and right, above and below, in front and behind, as well as the concepts of horizontal and vertical. As a consequence of this, the orthogonal borders of a picture lead the artist to the use of right-angled structures in the picture which arc then interpreted as the axes of Euclidean space.



Inspired by these ideas, Glattfelder created his series of ‚non-Euclidean metaphors‘, each of which consists of a polygon covered by a net of straight, coloured lines; for example Polarita secondaria, shown above, or Triplex, shown on p3O. His metaphors are an attempt to change our conventional habits of viewing our surroundings and, in this sense, they may be regarded as anti-perspectives. The straight lines induce a spatial, perspective illusion and appear to be curved. Glattfelder asks: “Can a straight line be curved at the same time?” The analysis shows, however, that the lines do not have a common focal point but form the tangents of an envelope.

The works of the constructive artists provide immense material for mathematical investigations and practical activities, as shown for the case of 2-D representations of 3-D space. There is no doubt that by discovering the mathematical structures in works of art, we are able to understand these works more fully and will come to a deeper appreciation of art in general.

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References

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2 B A R Carter, ‚Perspective‘, in: H Osborne (ed), The Oxford companion to art, Oxford, 1970, p 840-861
3 M H Pirenne, Qptics, painting & photography, Cambridge, University Press, 1970
4 E H Gombrich, Art and illusion, Phaidon Press, 1977
5 V Vasarely, Gaa, Herder, Freiburg, 1983
6 M Jammer, Concepts of space: the history of theories of space in physies, Harvard University Press, 1954

Illustrations

The illustrations on p30 are as follows, reading from right to left from the top of the page:

1 H P Reuter, Stadtbad ohne Ding 89-91, 1975, oil on canvas,250 x 180cm
2 M C Escher, Ascending and descending, 1960, lithograph,35x28.5 cm
3 J Albers, Structured constellation B-8, 1954, engraving in resopal, 43.5x57.5 cm
4 H Glattfelder, Triplex, 1983, acrylic on canvas/wood,140x 140cm
5 V Vasarely, Xexa-domb, 1971-3, 222x192cm
6 V Vasarely, Stri-meu, 1973-5, 178x200cm

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